Optimal estimation of squeezing

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(Dated: February 1, 2008)

We present the optimal estimation of an unknown squeezing transformation of the radiation field. The optimal estimation is unbiased and is obtained by properly considering the degeneracy of the squeezing operator. For coherent input states, the r.m.s. of the estimation scales as $(2\sqrt{\bar{n}})^{-1}$ versus the average photon number $\bar{n}$, while it can be enhanced to $(2\bar{n})^{-1}$ by using displaced squeezed states.

PACS numbers:

Squeezed states are characterized by a phase-dependent redistribution of quantum fluctuations such that the dispersion in one of the two quadrature components of the field is reduced below the level set by the symmetric distribution of the vacuum state or a coherent state [1]. Such a property has been used to raise the sensitivity beyond the standard quantum limit [2, 3] and to enhance interferometric [4] and absorption measurements [5], along with optical imaging applications [3, 6].

Even though squeezed states have been studied extensively during the last three decades, the attention to the problem of estimating an unknown squeezing transformation is relatively recent, and very few results are known about it. The first attempt to quantify the accuracy limits imposed by quantum mechanics was presented in Ref. [7], in the case of squeezing in a fixed direction. Here, the squeezing transformations form a one-parameter group, and the estimation problem is closely similar to the problem of phase estimation [8, 9] (for this reason the name *hyperbolic phase estimation* has been also used). The basic idea underlying the estimation strategy is to find a measurement that projects the quantum state on the vectors that are canonically conjugated via Fourier transform to the eigenstates of the squeezing generator. However, as we will show in this Letter, the scheme of Ref. [7] is not optimal and is biased: neither the mean value nor the most likely one in the probability distribution coincide with the true value of the squeezing parameter. In other words, the estimation is biased, and the presence of such a bias suggests that the proposed scheme is not optimal.

More recently, the estimation of squeezing has been considered in connection with cloning [10]. In this case the unknown squeezing is estimated from a number of identical copies of the same unknown squeezed state. However, this approach does not work when only a single copy is available, and the problem of the bias and the optimality of the estimation is left open.

In this Letter, we will present the optimal estimation of an unknown squeezing transformation in a given direction, acting on an arbitrary state of the radiation field. This problem is the optimal estimation of squeezing in an experimental situation where a degenerate parametric amplifier is pumped by a strong coherent field with a fixed phase relation with the state to be amplified.

We will show that the optimal measurement is unbiased, provided that one properly takes into account the degeneracy of the squeezing operator. Due to such a degeneracy, the Fourier transform of the eigenstates of the squeezing operator is not uniquely defined, and, in order to obtain the best estimation strategy, one has to perform an optimization similar to that of phase estimation with degeneracy [9]. Accordingly, the optimal estimation of squeezing depends on the chosen initial state of the radiation field. Also, the optimization performed here is analogous to that of Ref. [11] in the case of estimation of rotations, namely it properly takes into account the equivalent representations of the group of parameters. In fact, the degeneracy of the squeezing operator corresponds to the presence of equivalent representations of the related one-parameter group.

We will derive our results in the framework of quantum estimation theory [12, 13], upon defining optimality as the minimization of the expected value of a given cost function, which quantifies the deviation of the estimated parameter from the true one. According to the minimax approach, the optimal estimation strategy will be the one that minimizes the maximum of the expected cost over all possible true values of the unknown squeezing parameter. In analogy with the class of cost functions introduced by Holevo [8, 13] for the problem of phase estimation, we introduce here a class of cost functions including a large number of optimality criteria, such as maximum likelihood, and maximum fidelity. We will show that our estimation strategy is optimal according to any function in such a class.

In the following, we consider a single-mode radiation field with bosonic operators a and $a^\dagger$, satisfying the canonical commutation relations $[a, a^\dagger] = 1$. The squeezing operator is defined as follows

$$S(r) = \exp\left[\frac{r}{2}\left(a^{\dagger 2} - a^2\right)\right] , \qquad (1)$$

where r is a real parameter. Given a pure state $|\psi\rangle$ of the radiation field, we want to find the optimal measurement that allows one to estimate the parameter r in the transformation $|\psi\rangle \longrightarrow S(r)|\psi\rangle$. In the quadrature representation $\psi(x) = \langle x|\psi\rangle$, where $|x\rangle$ denotes the Dirac-normalized eigenvector of the quadrature operator $X = (a + a^\dagger)/2$, the effect of squeezing on the wavefunction is given by $\psi(x) \longrightarrow e^{-r/2}\psi(e^{-r}x)$.

The squeezing operator can be written as $S(r) = e^{-irK}$, where K is the Hermitian operator $K = i(a^{\dagger 2} - a^2)/2$, that generates the one-parameter group of squeezing transformations. The spectrum of the generator K is the whole real line, and the eigenvalue equation reads

$$K|\mu, s\rangle = \mu|\mu, s\rangle , \qquad (2)$$

where $\mu \in \mathbb{R}$ is the eigenvalue, and s is a degeneracy index with two possible values ± 1. The explicit expression of the generalized eigenvectors of K in the quadrature representation is given by [14]

$$\langle x|\mu, s\rangle = \frac{1}{\sqrt{2\pi}} |x|^{i\mu - \frac{1}{2}} \theta(sx) , \qquad (3)$$

where $\theta(x)$ is the Heaviside step-function [$\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$]. The vectors $|\mu, s\rangle$ are orthogonal in the Dirac sense, namely $\langle \mu, r|\nu, s\rangle = \delta_{rs}\, \delta(\mu - \nu)$, and provide the resolution of the identity

$$\int_{-\infty}^{+\infty} d\mu\, \Pi_\mu = \mathbb{1} , \qquad (4)$$

where $\Pi_\mu = \sum_{s=\pm 1} |\mu, s\rangle\langle \mu, s|$ is the projector onto the eigenspace of K corresponding to the eigenvalue μ.

Let us denote by $\mathcal{H}_\mu$ the two-dimensional vector space spanned by $|\mu, \pm 1\rangle$. In this complex vector space, we can consider the usual scalar product and the corresponding norm, namely if $|v_\mu\rangle = \sum_{s=\pm 1} v_s^\mu\, |\mu, s\rangle$ is an element of $\mathcal{H}_\mu$, then its norm is $|||v_\mu\rangle|| = \left(\sum_{s=\pm 1} |v_s^\mu|^2\right)^{1/2}$. Using the completeness relation (4), we can write any pure state $|\psi\rangle \in \mathcal{H}$ as

$$|\psi\rangle = \int_{-\infty}^{+\infty} d\mu\, c_\mu |\psi_\mu\rangle , \qquad (5)$$

where $c_\mu = ||\Pi_\mu|\psi\rangle||$, and

$$|\psi_\mu\rangle = \frac{\Pi_\mu|\psi\rangle}{||\Pi_\mu|\psi\rangle||} \qquad (6)$$

is the normalized projection of $|\psi\rangle$ onto $\mathcal{H}_\mu$. The representation of a state as in Eq. (5) corresponds to the fact that the Hilbert space $\mathcal{H}$ can be decomposed as a *direct integral* $\mathcal{H} = \int_{-\infty}^{+\infty} d\mu\, \mathcal{H}_\mu$. In this representation the effect of a squeezing transformation is given by

$$S(r)|\psi\rangle = \int_{-\infty}^{+\infty} d\mu\, c_\mu e^{-ir\mu}\, |\psi_\mu\rangle , \qquad (7)$$

i.e. the squeezing operator introduces a different phase shift in any space $\mathcal{H}_\mu$. Notice that the states (7) all lie in the subspace

$$\mathcal{H}_\psi = \left\{ |v\rangle = \int_{-\infty}^{+\infty} d\mu\, v_\mu |\psi_\mu\rangle \;\middle|\; v_\mu \in L^2(\mathbb{R}) \right\} . \qquad (8)$$

The problem of squeezing estimation in the representation (7) becomes formally equivalent to the problem of phase estimation.

In order to optimize the estimation of squeezing, we describe the estimation procedure with a *positive operator valued measure* (POVM) $P(\hat{r})$. The probability distribution of estimating $\hat{r}$ when the true value of squeezing is r is then given by $p(\hat{r}|r) = \mathrm{Tr}[P(\hat{r}) S_r \rho S_r^\dagger]$. The optimality criterion is specified in terms of a *cost function* $c(\hat{r} - r)$, that quantifies the cost of estimating $\hat{r}$ when the true value is r. Once a cost function has been fixed, the optimal measurement is defined in the minimax approach as the one that minimizes the quantity

$$\bar{c} = \max_{r \in \mathbb{R}} \left\{ \int_{-\infty}^{+\infty} d\hat{r}\, p(\hat{r}|r) c(\hat{r} - r) \right\} , \qquad (9)$$

namely the maximum of the expected cost over all possible true values. Generalizing the class of cost functions introduced by Holevo for phase estimation [13], we consider here cost functions of the form

$$c(r) = \int_0^{+\infty} d\mu\, a_\mu \cos(\mu r) , \qquad (10)$$

where $a_\mu \leq 0$ for any $\mu > 0$. This class contains a large number of optimality criteria, such as the maximum likelihood $c_{ML}(r) = -\delta(r)$, and the maximum fidelity $c_F(r) = 1 - |\langle\psi|S(r)|\psi\rangle|^2$.

Due to the group symmetry of the problem, instead of searching among all possible measurements for optimization, one can restrict attention to the class of *covariant* measurements [13], which are described by POVMs of the form $P(\hat{r}) = S(\hat{r})\xi S(\hat{r})^\dagger$, with $\xi \geq 0$ such that

$$\int_{-\infty}^{+\infty} dr\, S(r)\xi S(r)^\dagger = \mathbb{1} . \qquad (11)$$

The probability distribution $p(\hat{r}|r)$ related to a covariant measurement will depend only on the difference $\hat{r} - r$, and this means that the estimation is equally good for any possible value of the unknown squeezing [15].

The optimization of the covariant measurement for any cost function in the class (10) can be obtained as in the

case of phase estimation with degeneracy [9]. The optimal covariant POVM is then given by

$$P(r) = |\eta(r)\rangle\langle\eta(r)| \ , \tag{12}$$

where

$$|\eta(r)\rangle = \int_{-\infty}^{+\infty} \frac{d\mu}{\sqrt{2\pi}}\, e^{-ir\mu}\, |\psi_\mu\rangle \ . \tag{13}$$

Notice the correspondence of $|\eta(r)\rangle$ with the vectors $|e(\varphi)\rangle = \sum_{n=0}^{\infty} \frac{e^{in\varphi}}{\sqrt{2\pi}} |n\rangle$ that arise in the context of optimal phase estimation (here $|n\rangle$ are the non-degenerate eigenvectors of the photon number operator $a^\dagger a$). The vectors $|\eta(r)\rangle$ are orthogonal in the Dirac sense, namely the optimal POVM is a von Neumann measurement. The projection $|\psi_\mu\rangle$ of Eq. (6) in the expression of $|\eta(r)\rangle$ makes the optimal measurement depend on the input state $|\psi\rangle$. Accordingly, one obtains non-commuting observables, corresponding to different input states. The normalization of the POVM (12) can be easily checked, since $\int_{-\infty}^{+\infty} dr\ P(r) = 1\!\!1_\psi$, where $1\!\!1_\psi$ is the identity in the subspace $\mathcal{H}_\psi$ defined in Eq. (8). Clearly, the $P(r)$ can be arbitrarily completed to the whole Hilbert space, without affecting the probability distribution of the outcomes.

Using Eq. (6), the optimal probability distribution for an input state $|\psi\rangle$ is given by

$$\begin{aligned} p(\hat r|r) &= \langle\psi|S(r)^\dagger\ P(\hat r)\ S(r)|\psi\rangle \\ &= \frac{1}{2\pi}\left|\int_{-\infty}^{+\infty} d\mu\, e^{-i(\hat r - r)\mu}\sqrt{\langle\psi|\Pi_\mu|\psi\rangle}\right|^2 . \end{aligned} \tag{14}$$

Since the probability distribution depends only on the difference $\hat r - r$, from now on we will write $p(\hat r - r)$ instead of $p(\hat r|r)$. Representing the projection Π_μ as $\Pi_\mu = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi}\, e^{i\lambda(\mu - K)}$, the probability distribution of Eq. (14) can be rewritten as

$$p(r) = \left|\int_{-\infty}^{+\infty} \frac{d\mu}{2\pi}\, e^{-ir\mu}\sqrt{\int_{-\infty}^{+\infty} d\lambda\, e^{i\lambda\mu}\,\langle\psi|S(\lambda)|\psi\rangle}\right|^2 \tag{15}$$

The optimal measurement (12) can be compared with that given in Ref. [7], which is described in our notation by the POVM

$$\tilde P(r) = \sum_{s=\pm 1} |\eta_s(r)\rangle\langle\eta_s(r)| \ , \tag{16}$$

where

$$|\eta_s(r)\rangle = \int_{-\infty}^{+\infty} \frac{d\mu}{\sqrt{2\pi}}\ e^{-ir\mu}|\mu, s\rangle \ . \tag{17}$$

Using Eq. (3), it is easy to see that $|\eta_\pm(r)\rangle$ are eigenvectors of the quadrature X corresponding to the eigenvalues $\pm e^r$, and hence the POVM (16) corresponds to measuring the observable $\ln|X|$, *independently* of the input state. The measurement $\tilde P(r)$ is not optimal, and gives a biased probability distribution, namely the average value of the estimated parameter does not coincide with the true value, and also the most likely value in the probability distribution is not the true one (see, e.g., the asymmetric probability distribution for the vacuum state in Fig. 1). Such drawbacks do not occur in the optimal probability distribution (14). Notice also that the measurement $\tilde P(r)$ is "rank-two" in the subspace $\mathcal{H}_\psi$ of interest, while the optimal measurement is "rank-one". The differences between the two measurements can be understood intuitively as follows. Essentially, both POVMs are based on the Fourier transform of the eigenvectors of the operator K. However, since the Fourier transform is not uniquely defined due to the degeneracy of K, one should optimize it versus the input state.

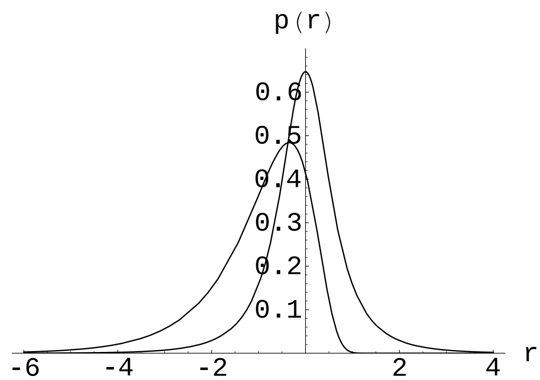

FIG. 1: Probability distributions for the estimation of squeezing on a vacuum input state. The asymmetric distribution comes from the suboptimal measurement of Ref. [7] in Eq. (16). The symmetric distribution corresponds to the optimal measurement of Eq. (12).

In the case of a coherent input state $|\alpha\rangle$, the probability distribution (15) can be specified as follows

$$p(r) = e^{-|\alpha|^2}\left|\int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}\, e^{-i\mu r}\ \times \sqrt{\int_{-\infty}^{+\infty}\frac{d\lambda}{\sqrt{\cosh\lambda}}\, e^{i\lambda\mu}\, e^{\frac{1}{2}\tanh\lambda(\alpha^{*2}-\alpha^2)}\, e^{\frac{|\alpha|^2}{\cosh\lambda}}}\right|^2 \tag{18}$$

This probability distribution has been plotted for increasing real values of α in Fig. 2, where one can easily observe the corresponding improvement in the estimation.

For large values of $|\alpha|$, from Eq. (19) one obtains asymptotically the Gaussian distribution

$$p(r) = \sqrt{\frac{2|\alpha|^2}{\pi}}\, e^{-2|\alpha|^2 r^2} \ , \tag{19}$$

that provides a r.m.s error on the estimation of r as $\Delta r = 1/(2\sqrt{\bar n})$, where $\bar n = |\alpha|^2$ is the mean photon number. This scaling can be improved to $\Delta r = 1/(2\bar n)$

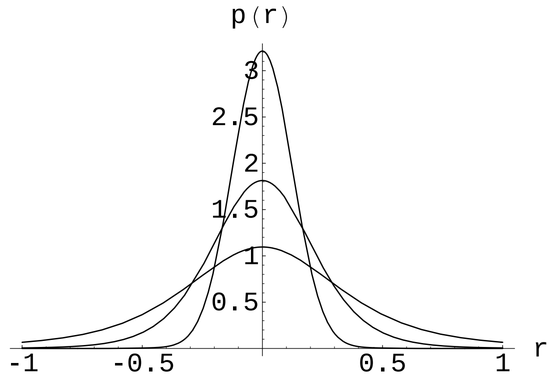

FIG. 2: Optimal probability distribution of squeezing for input coherent states. The distribution becomes sharper for increasing values of the coherent-state amplitude ($\alpha = 1$, 2, 4.)

by using displaced squeezed states $|\alpha, z\rangle = D(\alpha)S(z)|0\rangle$, with $\alpha, z \in \mathbb{R}$. In fact, from the relation $D(\alpha)S(z) = S(z)D(\alpha e^{-z})$, the probability distribution $p(r)$ is given by Eq. (18) just by replacing α with αe^{-z}. In the asymptotic limit of large number of photons $\bar{n} = |\alpha|^2 + \sinh^2 z$, the minimization of the r.m.s. gives the optimal scaling $\Delta r = 1/(2\bar{n})$, for $\alpha = \sqrt{\bar{n}/2}$ and $z = -1/2 \ln(2\bar{n})$, and this corresponds to approximate the eigenvectors of the quadrature operator X.

In the asymptotic regime, the optimal performance can be achieved simply by measuring the quadrature X and estimating $\hat{r} = \ln |x/\alpha|$, in correspondence to the outcome x. However, it is important to stress that homodyne measurement becomes optimal only for particular input states and in the asymptotic limit of large energy, while for finite energy the optimal measurement is described by the POVM in Eq. (12).

In conclusion, we presented the covariant measurement for estimating the squeezing that is optimal for a large class of figure of merit. The optimal detection is given by a suitable Fourier transform of the eigenstates of the generator of squeezing. In fact, due to the degeneracy of the squeezing operator, there is a freedom in choosing how to perform the Fourier transform, and the choice must be optimized according to the input state. Hence, for different input states one has different optimal estimations corresponding to different observables. The optimal measurement leads to an unbiased estimation, and the outcome of the measurement that is most likely to be obtained coincides with the true value of the unknown squeezing. For coherent input states the r.m.s. error scales as $1/(2\sqrt{\bar{n}})$ with the number of photons, while for displaced squeezed states one achieves $1/(2\bar{n})$ scaling. In the asymptotic regime, such a scaling can be obtained experimentally by homodyne measurement. The presented scheme applies to the problem of optimal characterization of nondegenerate parametric amplifiers.

We acknowledge M. Hayashi for suggesting Ref. [16]. This work has been supported by Ministero Italiano dell'Università e della Ricerca (MIUR) through FIRB (bando 2001) and PRIN 2005.

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